

# Nonlinear Spinor Fields and its role in Cosmology

Bijan Saha

*Laboratory of Information Technologies  
Joint Institute for Nuclear Research  
141980 Dubna, Moscow region, Russia\**

Different characteristic of matter influencing the evolution of the Universe has been simulated by means of a nonlinear spinor field. Exploiting the spinor description of perfect fluid and dark energy evolution of the Universe given by an anisotropic Bianchi type-VI, VI<sub>0</sub>, V, III, I or isotropic Friedmann-Robertson-Walker (FRW) one has been studied. It is shown that due to some restrictions on metric functions, initial anisotropy in the models Bianchi type-VI, VI<sub>0</sub>, V and III does not die away, while the anisotropic Bianchi type-I models evolves into the isotropic one.

PACS numbers: 98.80.Cq

Keywords: Spinor field, perfect fluid, dark energy, anisotropic cosmological models

## I. INTRODUCTION

Being related to almost all stable elementary particles such as proton, electron and neutrino, spinor field, especially Dirac spin-1/2 play a principal role at the microlevel. However, in cosmology, the role of spinor field was generally considered to be restricted. Only recently, after some remarkable works by different authors (e.g., Henneaux, 1980; Saha, 2001a, 2004a, 2004b, 2006c, 2006d; Armendáriz-Picón and Greene, 2003; Ribas *et al.*, 2005; Souza and Kremer, 2008), showing the important role that spinor fields play on the evolution of the Universe, the situation began to change. This change of attitude is directly related to some fundamental questions of modern cosmology: (i) problem of initial singularity; (ii) problem of isotropization and (iii) late time acceleration of the Universe.

**(i) Problem of initial singularity:** One of the problems of modern cosmology is the presence of initial singularity, which means the finiteness of time. The main purpose of introducing a nonlinear term in the spinor field Lagrangian is to study the possibility of the elimination of initial singularity. In a number of papers it was shown that the introduction of nonlinear spinor field into the system indeed gives rise to singularity-free models of the Universe (Saha, 2001a, 2001b, 2004a), (Saha and Shikin, 1997a, 1997b).

**(ii) problem of isotropization:** Although the Universe seems homogenous and isotropic at present, it does not necessarily mean that it is also suitable for a description of the early stages of the development of the Universe and there are no observational data guaranteeing the isotropy in the era prior to the recombination. In fact, there are theoretical arguments that support the existence of an anisotropic phase that approaches an isotropic one (Misner 1968). The observations from Cosmic Background Explorer's differential radiometer have detected and measured cosmic microwave background anisotropies in different angular scales. These anisotropies are supposed to hide in their fold the entire history of cosmic evolution dating back to the recombination era and are being considered as indicative of the geometry and the content of the universe. More about cosmic microwave background anisotropy is expected to be uncovered by the investigations of microwave anisotropy probe. There is widespread consensus among the cosmologists that cosmic microwave background anisotropies in small angular scales have the key to the formation of discrete structure.

---

\*Electronic address: bijan@jinr.ru; URL: <http://bijansaha.narod.ru>

It was found that the introduction of nonlinear spinor field accelerates the isotropization process of the initially anisotropic Universe (Saha, 2001a, 2004a, 2006c).

**(iii) late time acceleration of the Universe:** Detection and further experimental reconfirmation of current cosmic acceleration pose to cosmology a fundamental task of identifying and revealing the cause of such phenomenon. This fact can be reconciled with the theory if one assumes that the Universe is mostly filled with so-called dark energy. This form of matter (energy) is not observable in laboratory and it does not interact with electromagnetic radiation. These facts played decisive role in naming this object. In contrast to dark matter, dark energy is uniformly distributed over the space, does not intertwine under the influence of gravity in all scales and it has a strong negative pressure of the order of energy density. Based on these properties, cosmologists have suggested a number of dark energy models, those are able to explain the current accelerated phase of expansion of the Universe. In this connection a series of papers appeared recently in the literature, where a spinor field was considered as an alternative model for dark energy (Ribas *et al*, 2005; Saha, 2006d, 2006e, 2007).

It should be noted that most of the works mentioned above were carried out within the scope of Bianchi type-I cosmological model. Results obtained using a spinor field as a source of Bianchi type-I cosmological field can be summed up as follows: A suitable choice of spinor field nonlinearity

- (i) *accelerates the isotropization process* (Saha, 2001a, 2004a, 2006c);
- (ii) *gives rise to a singularity-free Universe* (Saha, 2001a, 2004a, 2004b, 2006c);
- (iii) *generates late time acceleration* (Ribas *et al*, 2005; Saha, 2006d; Souza and Kremer, 2008).

Given the role that spinor field can play in the evolution of the Universe, question that naturally pops up is, if the spinor field can redraw the picture of evolution caused by perfect fluid and dark energy, is it possible to simulate perfect fluid and dark energy by means of a spinor field? Affirmative answer to this question was given in a number of papers (Krechet *et al*, 2008; Saha, 2010a, 2010b). In those papers the authors have shown that different types of perfect fluid and dark energy can be described by nonlinear spinor field. In (Saha, 2010a) we used two types of nonlinearity, one occurs as a result of self-action and the other resulted from the interaction between the spinor and scalar field. It was shown that the case with induced nonlinearity is the partial one and can be derived from the case with self-action. In (Saha, 2010b, 2011) we give the description of generalized Chaplygin gas and modified quintessence in terms of spinor field and study the evolution of the Universe filled with nonlinear spinor field within the scope of a Bianchi type-I and FRW cosmological model. The purpose of this paper is to extend that study within the framework of other Bianchi models.

## II. SIMULATION OF PERFECT FLUID WITH NONLINEAR SPINOR FIELD

Nonlinear quantum Dirac fields were used by Heisenberg (1953, 1957) in his ambitious unified theory of elementary particles. They are presently the object of renewed interest since the widely known paper by Gross and Neveu (1974). A nonlinear spinor field, suggested by the symmetric coupling between nucleons, muons, and leptons, has been investigated by Finkelstein *et al*. (1951) in the classical approximation.

In this section we simulate different types of perfect fluid and dark energy by means of a nonlinear spinor field.

### A. Spinor field Lagrangian

For a spinor field  $\psi$ , the symmetry between  $\psi$  and  $\bar{\psi}$  appears to demand that one should choose the symmetrized Lagrangian (Kibble, 1961). Keeping this in mind we choose the spinor field Lagrangian as (Saha, 2001a):

$$L_{sp} = \frac{i}{2} \left[ \bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi \right] - m_{sp} \bar{\psi} \psi + F, \quad (2.1)$$

where the nonlinear term  $F$  describes the self-action of a spinor field and can be presented as some arbitrary functions of invariant generated from the real bilinear forms of a spinor field. For simplicity we consider the case when  $F = F(S)$  with  $S = \bar{\psi}\psi$ . Here  $\nabla_\mu$  is the covariant derivative of spinor field:

$$\nabla_\mu \psi = \frac{\partial \psi}{\partial x^\mu} - \Gamma_\mu \psi, \quad \nabla_\mu \bar{\psi} = \frac{\partial \bar{\psi}}{\partial x^\mu} + \bar{\psi} \Gamma_\mu, \quad (2.2)$$

with  $\Gamma_\mu$  being the spinor affine connection. In (2.1)  $\gamma$ 's are the Dirac matrices in curve space-time and obey the following algebra

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \quad (2.3)$$

and are connected with the flat space-time Dirac matrices  $\bar{\gamma}$  in the following way

$$g_{\mu\nu}(x) = e_\mu^a(x) e_\nu^b(x) \eta_{ab}, \quad \gamma_\mu(x) = e_\mu^a(x) \bar{\gamma}_a, \quad (2.4)$$

where  $\eta_{ab} = \text{diag}(1, -1, -1, -1)$  and  $e_\mu^a$  is a set of tetrad 4-vectors. The spinor affine connection matrices  $\Gamma_\mu(x)$  are uniquely determined up to an additive multiple of the unit matrix by the equation

$$\nabla_\mu \gamma_\nu = \frac{\partial \gamma_\nu}{\partial x^\mu} - \Gamma_{\nu\mu}^\rho \gamma_\rho - \Gamma_\mu \gamma_\nu + \gamma_\nu \Gamma_\mu = 0, \quad (2.5)$$

with the solution

$$\Gamma_\mu = \frac{1}{4} \bar{\gamma}_a \gamma^\nu \partial_\mu e_\nu^{(a)} - \frac{1}{4} \gamma_\rho \gamma^\nu \Gamma_{\mu\nu}^\rho, \quad (2.6)$$

Varying (2.1) with respect to  $\bar{\psi}(\psi)$  one finds the spinor field equations:

$$i\gamma^\mu \nabla_\mu \psi - m_{\text{sp}} \psi + \frac{dF}{dS} \psi = 0, \quad (2.7a)$$

$$i\nabla_\mu \bar{\psi} \gamma^\mu + m_{\text{sp}} \bar{\psi} - \frac{dF}{dS} \bar{\psi} = 0, \quad (2.7b)$$

The energy-momentum tensor of the spinor field is given by

$$T_\mu^\rho = \frac{i}{4} g^{\rho\nu} \left( \bar{\psi} \gamma_\mu \nabla_\nu \psi + \bar{\psi} \gamma_\nu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma_\nu \psi - \nabla_\nu \bar{\psi} \gamma_\mu \psi \right) - \delta_\mu^\rho L_{\text{sp}} \quad (2.8)$$

where  $L_{\text{sp}}$  in account of spinor field equations (2.7a) and (2.7b) takes the form

$$L_{\text{sp}} = -S \frac{dF}{dS} + F(S). \quad (2.9)$$

We consider the case when the spinor field depends on  $t$  only. In this case for the components of energy-momentum tensor we find

$$T_0^0 = m_{\text{sp}} S - F, \quad (2.10a)$$

$$T_1^1 = T_2^2 = T_3^3 = S \frac{dF}{dS} - F. \quad (2.10b)$$

A detailed study of nonlinear spinor field was carried out in Saha (2001a, 2004a, 2006c). In what follows, exploiting the equation of states we find the concrete form of  $F$  which describes various types of perfect fluid and dark energy.

### B. perfect fluid with a barotropic equation of state

First of all let us note that one of the simplest and popular model of the Universe is a homogeneous and isotropic one filled with a perfect fluid with the energy density  $\varepsilon = T_0^0$  and pressure  $p = -T_1^1 = -T_2^2 = -T_3^3$  obeying the barotropic equation of state

$$p = W\varepsilon, \quad (2.11)$$

where  $W$  is a constant. Depending on the value of  $W$  (2.11) describes perfect fluid from phantom to ekpyrotic matter, namely

$$W = 0, \quad (\text{dust}), \quad (2.12a)$$

$$W = 1/3, \quad (\text{radiation}), \quad (2.12b)$$

$$W \in (1/3, 1), \quad (\text{hard Universe}), \quad (2.12c)$$

$$W = 1, \quad (\text{stiff matter}), \quad (2.12d)$$

$$W \in (-1/3, -1), \quad (\text{quintessence}), \quad (2.12e)$$

$$W = -1, \quad (\text{cosmological constant}), \quad (2.12f)$$

$$W < -1, \quad (\text{phantom matter}), \quad (2.12g)$$

$$W > 1, \quad (\text{ekpyrotic matter}). \quad (2.12h)$$

The barotropic model of perfect fluid is used to study the evolution of the Universe. Most recently the relation (2.11) is exploited to generate a quintessence in order to explain the accelerated expansion of the Universe (Saha, 2005; Zlatev 1999).

In order to describe the matter given by (2.12) with a spinor field let us now substitute  $\varepsilon$  and  $p$  with  $T_0^0$  and  $-T_1^1$ , respectively. Thus, inserting  $\varepsilon = T_0^0$  and  $p = -T_1^1$  from (2.10a) and (2.10b) into (2.11) we find

$$S \frac{dF}{dS} - (1 + W)F + m_{\text{sp}}WS = 0, \quad (2.13)$$

with the solution (Saha, 2010a, 2010b, 2011)

$$F = \lambda S^{1+W} + m_{\text{sp}}S. \quad (2.14)$$

Here  $\lambda$  is an integration constant. Taking into account that the energy density should be non-negative, we conclude that  $\lambda$  is a negative constant, we write the components of the energy momentum tensor

$$T_0^0 = \nu S^{1+W}, \quad (2.15a)$$

$$T_1^1 = T_2^2 = T_3^3 = -\nu W S^{1+W}, \quad (2.15b)$$

where  $\lambda = -\nu$ , with  $\nu$  being a positive constant. As one sees, the energy density  $\varepsilon = T_0^0$  is always positive, while the pressure  $p = -T_1^1 = \nu W S^{1+W}$  is positive for  $W > 0$ , i.e., for usual fluid and negative for  $W < 0$ , i.e. for dark energy.

In account of it the spinor field Lagrangian now reads

$$L_{\text{sp}} = \frac{i}{2} \left[ \bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi \right] - \nu S^{1+W}, \quad (2.16)$$

Thus a massless spinor field with the Lagrangian (2.16) describes perfect fluid from phantom to ekpyrotic matter. Here the constant of integration  $\nu$  can be viewed as constant of self-coupling. A detailed analysis of this study was given in Krechet (2008).

### C. Chaplygin gas

An alternative model for the dark energy density was used by Kamenshchik *et al.* (2001), where the authors suggested the use of some perfect fluid but obeying "exotic" equation of state. This type of matter is known as *Chaplygin gas*. The fate of density perturbations in a Universe dominated by the Chaplygin gas, which exhibit negative pressure was studied by Fabris *et al.* (2002). Model with Chaplygin gas was also studied in the Refs. (Dev *et al.*, 2003; Bento *et al.*, 2002).

Let us now generate a Chaplygin gas by means of a spinor field. A Chaplygin gas is usually described by a equation of state

$$p = -A/\varepsilon^\gamma. \quad (2.17)$$

Then in case of a massless spinor field for  $F$  one finds

$$\frac{(-F)^\gamma d(-F)}{(-F)^{1+\gamma} - A} = \frac{dS}{S}, \quad (2.18)$$

with the solution (Saha, 2010a, 2010b, 2011)

$$-F = (A + \lambda S^{1+\gamma})^{1/(1+\gamma)}. \quad (2.19)$$

On account of this for the components of energy momentum tensor we find

$$T_0^0 = (A + \lambda S^{1+\gamma})^{1/(1+\gamma)}, \quad (2.20a)$$

$$T_1^1 = T_2^2 = T_3^3 = A/(A + \lambda S^{1+\gamma})^{\gamma/(1+\gamma)}. \quad (2.20b)$$

As was expected, we again get positive energy density and negative pressure. Analogical results were obtained in (Cai *et al.*, 2008).

Thus the spinor field Lagrangian corresponding to a Chaplygin gas reads

$$L_{\text{sp}} = \frac{i}{2} \left[ \bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi \right] - (A + \lambda S^{1+\gamma})^{1/(1+\gamma)}. \quad (2.21)$$

Setting  $\gamma = 1$  we find the result obtained in Saha (2010a).

### D. Modified quintessence

Finally, we simulate modified quintessence with a nonlinear spinor field. It should be noted that one of the problems that face models with dark energy is that of eternal acceleration. One of the possible way to avoid this problem is to introduce a negative  $\Lambda$ -term together with a quintessence, which gives rise to a oscillatory mode of expansion (Cardenas *et. al.*, 2003, Saha, 2006a) In order to get rid of that problem quintessence with a modified equation of state was proposed which is given by (Saha, 2006b)

$$p = W(\varepsilon - \varepsilon_{\text{cr}}), \quad W \in (-1, 0), \quad (2.22)$$

Here  $\varepsilon_{\text{cr}}$  some critical energy density. Setting  $\varepsilon_{\text{cr}} = 0$  one obtains ordinary quintessence. It is well known that as the Universe expands the (dark) energy density decreases. As a result, being a linear negative function of energy density, the corresponding pressure begins to increase. In case of an ordinary quintessence the pressure is always negative, but for a modified quintessence as soon as  $\varepsilon$  becomes less than the critical one, the pressure becomes positive.

Inserting  $\varepsilon = T_0^0$  and  $p = -T_1^1$  into (2.22) we find

$$F = -vS^{1+W} + m_{\text{sp}}S - \frac{W}{1+W}\varepsilon_{\text{cr}}, \quad (2.23)$$

with  $\eta$  being a positive constant. On account of this for the components of energy momentum tensor we find

$$T_0^0 = vS^{1+W} + \frac{W}{1+W}\epsilon_{\text{cr}}, \quad (2.24a)$$

$$T_1^1 = T_2^2 = T_3^3 = -vWS^{1+W} + \frac{W}{1+W}\epsilon_{\text{cr}}. \quad (2.24b)$$

Lagrangian for spinor field describing perfect fluid and modified quintessence can be written in the following way (Saha, 2011)

$$L_{\text{sp}} = \frac{i}{2} \left[ \bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi \right] - vS^{1+W} - \frac{W}{1+W}\epsilon_{\text{cr}}. \quad (2.25)$$

One can easily verify, in case of  $\epsilon_{\text{cr}} = 0$  (2.25) corresponds to a perfect fluid, while nontrivial  $\epsilon_{\text{cr}}$  with  $W \in (-1, 0)$  generates modified quintessence. It should be noted that the restriction  $W > -1$  is very important, as it does not allow the system to cross over phantom divide barrier.

We see that a nonlinear spinor field with specific type of nonlinearity can substitute perfect fluid and dark energy, thus give rise to a variety of evolution scenario of the Universe.

### III. COSMOLOGICAL MODELS WITH A SPINOR FIELD

In the previous section we showed that the perfect fluid and the dark energy can be simulated by a nonlinear spinor field. In the section II the nonlinearity was the subject to self-action. In (Saha, 2010a) we have also considered the case when the nonlinearity was induced by a scalar field. It was also shown the in our context the results for induced nonlinearity is some special cases those of self-interaction. Taking it into mind we study the evolution an Universe filled with a nonlinear spinor field given by the Lagrangian (2.1), with the nonlinear term  $F$  is given by (2.14), (2.19) and (2.23).

#### A. Bianchi type anisotropic cosmological model

Let us study the evolution of an anisotropic Bianchi type cosmological model filled with spinor field. In this report we consider Bianchi type-VI, VI<sub>0</sub>, V, III, I and FRW models.

We choose the Bianchi type-VI cosmological model in the form (Saha, 2004b)

$$ds^2 = dt^2 - a_1^2 e^{-2mz} dx^2 - a_2^2 e^{2nz} dy^2 - a_3^2 dz^2, \quad (3.1)$$

with  $a_1, a_2, a_3$  being the function of  $t$  only. A suitable choice of  $m, n$  as well as the metric functions  $a_1, a_2, a_3$  in the BVI given by (3.1) evokes the following Bianchi-type universes:

- for  $m = n$  the BVI metric transforms to a Bianchi-type VI<sub>0</sub> (BVI<sub>0</sub>) one, i.e.,  $m = n$ , BVI  $\implies$  BVI<sub>0</sub>  $\in$  open FRW with the line elements

$$ds^2 = dt^2 - a_1^2 e^{-2mz} dx^2 - a_2^2 e^{2mz} dy^2 - a_3^2 dz^2; \quad (3.2)$$

- for  $m = -n$  the BVI metric transforms to a Bianchi-type V (BV) one, i.e.,  $m = n$ , BVI  $\implies$  BV  $\in$  open FRW with the line elements

$$ds^2 = dt^2 - a_1^2 e^{2mz} dx^2 - a_2^2 e^{2mz} dy^2 - a_3^2 dz^2; \quad (3.3)$$

- for  $n = 0$  the BVI metric transforms to a Bianchi-type III (BIII) one, i.e.,  $n = 0$ ,  $\text{BVI} \implies \text{BIII}$  with the line elements

$$ds^2 = dt^2 - a_1^2 e^{-2mz} dx^2 - a_2^2 dy^2 - a_3^2 dz^2; \quad (3.4)$$

- for  $m = n = 0$  the BVI metric transforms to a Bianchi-type I (BI) one, i.e.,  $m = n = 0$ ,  $\text{BVI} \implies \text{BI}$  with the line elements

$$ds^2 = dt^2 - a_1^2 dx^2 - a_2^2 dy^2 - a_3^2 dz^2; \quad (3.5)$$

- for  $m = n = 0$  and equal scale factor in all three directions the BVI metric transforms to a Friedmann-Robertson-Walker (FRW) universe, i.e.,  $m = n = 0$  and  $a = b = c$ ,  $\text{BVI} \implies \text{FRW}$  with the line elements.

$$ds^2 = dt^2 - a^2 (dx^2 + dy^2 + dz^2). \quad (3.6)$$

Let us go back to the metric Bianchi-type VI. The metric (3.1) possesses following nontrivial covariant and contravariant components:

$$\begin{aligned} g_{00} &= 1, & g_{11} &= a_1^2 e^{-2mz}, & g_{22} &= a_2^2 e^{2nz}, & g_{33} &= a_3^2, \\ g^{00} &= 1, & g^{11} &= \frac{1}{a_1^2 e^{-2mz}}, & g^{22} &= \frac{1}{a_2^2 e^{2nz}}, & g^{33} &= \frac{1}{a_3^2}. \end{aligned}$$

The nontrivial Christoffel symbols for (3.1) are

$$\begin{aligned} \Gamma_{01}^1 &= \frac{\dot{a}_1}{a_1}, & \Gamma_{02}^2 &= \frac{\dot{a}_2}{a_2}, & \Gamma_{03}^3 &= \frac{\dot{a}_3}{a_3}, \\ \Gamma_{11}^0 &= a_1 \dot{a}_1 e^{-2mz}, & \Gamma_{22}^0 &= a_2 \dot{a}_2 e^{2nz}, & \Gamma_{33}^0 &= a_3 \dot{a}_3, \\ \Gamma_{31}^1 &= -m, & \Gamma_{32}^2 &= n, & \Gamma_{11}^3 &= \frac{ma_1^2}{a_3^2} e^{-2mz}, & \Gamma_{22}^3 &= -\frac{na_2^2}{a_3^2} e^{2nz}. \end{aligned} \quad (3.7)$$

In view of (2.4) we choose the tetrad as follows:

$$e_0^{(0)} = 1, \quad e_1^{(1)} = a_1 e^{-mz}, \quad e_2^{(2)} = a_2 e^{nz}, \quad e_3^{(3)} = a_3. \quad (3.8)$$

From

$$\gamma_\mu = e_\mu^{(a)} \tilde{\gamma}_a. \quad (3.9)$$

one now finds

$$\gamma_0 = \tilde{\gamma}_0, \quad \gamma_1 = a_1 e^{-mz} \tilde{\gamma}_1, \quad \gamma_2 = a_2 e^{nz} \tilde{\gamma}_2, \quad \gamma_3 = a_3 \tilde{\gamma}_3. \quad (3.10)$$

Taking into account that in our case

$$\tilde{\gamma}^0 = \tilde{\gamma}_0, \quad \tilde{\gamma}^1 = -\tilde{\gamma}_1, \quad \tilde{\gamma}^2 = -\tilde{\gamma}_2, \quad \tilde{\gamma}^3 = -\tilde{\gamma}_3,$$

one also finds

$$\gamma^0 = \tilde{\gamma}^0, \quad \gamma^1 = \frac{e^{mz}}{a_1} \tilde{\gamma}^1, \quad \gamma^2 = \frac{e^{-nz}}{a_2} \tilde{\gamma}^2, \quad \gamma^3 = \frac{1}{a_3} \tilde{\gamma}^3. \quad (3.11)$$

Now we are ready to compute spinor affine connections using (2.6) which in our particular case gives

$$\Gamma_0 = \frac{1}{4}\bar{\gamma}_a\gamma^\nu\partial_t e^{(a)}_\nu - \frac{1}{4}\gamma_\rho\gamma^\nu\Gamma_{0\nu}^\rho = 0, \quad (3.12a)$$

$$\Gamma_1 = -\frac{1}{4}\gamma_\rho\gamma^\nu\Gamma_{1\nu}^\rho = \frac{1}{2}\left(\dot{a}_1\bar{\gamma}^1\bar{\gamma}^0 - m\frac{a_1}{a_3}\bar{\gamma}^1\bar{\gamma}^3\right)e^{-mz}, \quad (3.12b)$$

$$\Gamma_2 = -\frac{1}{4}\gamma_\rho\gamma^\nu\Gamma_{2\nu}^\rho = \frac{1}{2}\left(\dot{a}_2\bar{\gamma}^2\bar{\gamma}^0 + n\frac{a_2}{a_3}\bar{\gamma}^2\bar{\gamma}^3\right)e^{nz}, \quad (3.12c)$$

$$\Gamma_3 = \frac{1}{4}\bar{\gamma}_a\gamma^\nu\partial_z e^{(a)}_\nu - \frac{1}{4}\gamma_\rho\gamma^\nu\Gamma_{3\nu}^\rho = \frac{1}{2}\dot{a}_3\bar{\gamma}^3\bar{\gamma}^0. \quad (3.12d)$$

From (3.12) one finds

$$\gamma^\mu\Gamma_\mu = -\frac{1}{2}\left(\frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3}\right)\bar{\gamma}^0 + \frac{m-n}{2a_3}\bar{\gamma}^3. \quad (3.13)$$

Choosing  $m, n$  we can thus write the spinor affine connections for other Bianchi type metrics.

### 1. Bianchi type-VI anisotropic cosmological model

Let us now study the evolution of the Universe given by a Bianchi type-VI cosmological model. The Einstein equations corresponding to the metric (3.1) take the form:

$$\frac{\ddot{a}_2}{a_2} + \frac{\ddot{a}_3}{a_3} + \frac{\dot{a}_2\dot{a}_3}{a_2a_3} - \frac{n^2}{a_3^2} = \kappa T_1^1, \quad (3.14a)$$

$$\frac{\ddot{a}_3}{a_3} + \frac{\ddot{a}_1}{a_1} + \frac{\dot{a}_3\dot{a}_1}{a_3a_1} - \frac{m^2}{a_3^2} = \kappa T_2^2, \quad (3.14b)$$

$$\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1\dot{a}_2}{a_1a_2} + \frac{mn}{a_3^2} = \kappa T_3^3, \quad (3.14c)$$

$$\frac{\dot{a}_1\dot{a}_2}{a_1a_2} + \frac{\dot{a}_2\dot{a}_3}{a_2a_3} + \frac{\dot{a}_3\dot{a}_1}{a_3a_1} - \frac{m^2 - mn + n^2}{a_3^2} = \kappa T_0^0, \quad (3.14d)$$

$$m\frac{\dot{a}_1}{a_1} - n\frac{\dot{a}_2}{a_2} - (m-n)\frac{\dot{a}_3}{a_3} = \kappa T_3^0. \quad (3.14e)$$

Taking into account that we are dealing with massless spinor field, from (2.10a) and (2.10b) one finds  $T_0^0 = -F$ ,  $T_1^1 = T_2^2 = T_3^3 = S\frac{dF}{dS} - F$  and  $T_3^0 = 0$ . Then the Eq. (3.14e) immediately gives

$$\left(\frac{a_1}{a_3}\right)^m = \mathcal{N}_1\left(\frac{a_2}{a_3}\right)^n, \quad \mathcal{N}_1 = \text{const.} \quad (3.15)$$

Let us now define

$$V = a_1a_2a_3. \quad (3.16)$$

Summation of (3.14a), (3.14b), (3.14c) and three times (3.14d) gives

$$\frac{\ddot{V}}{V} = 2\frac{m^2 - mn + n^2}{a_3^2} + \frac{3\kappa}{2}\left[S\frac{dF}{dS} - 2F\right]. \quad (3.17)$$



Before solving this equation let us write  $S$  in terms of  $v$ . The spinor field equations in this case read

$$\bar{\gamma}^0 \left( \dot{\psi} + \frac{\dot{V}}{2V} \psi \right) - \frac{m-n}{2a_3} \bar{\gamma}^3 \psi - i \frac{dF}{dS} \psi = 0, \quad (3.18a)$$

$$\left( \dot{\bar{\psi}} + \frac{\dot{v}}{2v} \bar{\psi} \right) \bar{\gamma}^0 - \frac{m-n}{2a_3} \bar{\psi} \bar{\gamma}^3 + i \frac{dF}{dS} \bar{\psi} = 0. \quad (3.18b)$$

From (3.18) one finds

$$\frac{d(VS)}{dt} = 0, \quad (3.19)$$

which gives

$$S = \frac{C_0}{V}, \quad C_0 = \text{const.} \quad (3.20)$$

Thus we see that the equation for defining  $V$  explicitly depends on  $a_3$ . Here we assume that the expansion  $\theta$  is proportional to the eigenvalue  $\sigma_1^1$  of shear tensor  $\sigma_\mu^\nu$ . In a comoving system of reference with  $U^\mu = (1, 0, 0, 0)$  for Bianchi type-VI metric we find

$$\theta = U_{;\mu}^\mu = \frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3}. \quad (3.21)$$

From

$$\sigma_{\mu\nu} = \frac{1}{2} (U_{\mu;\rho} P_\nu^\rho + U_{\nu;\rho} P_\mu^\rho) - \frac{1}{3} \theta P_{\mu\nu}, \quad (3.22)$$

we find

$$\sigma_1^1 = -\frac{1}{3} \left( -2 \frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3} \right). \quad (3.23)$$

As one sees,  $S$ ,  $\theta$  and  $\sigma_1^1$  (it is true for  $\sigma_2^2$  and  $\sigma_3^3$  as well) do not depend on  $m$  and  $n$ . Hence, they will remain unaltered for other Bianchi models following from (3.1). Now setting

$$\sigma_1^1 = q_1 \theta, \quad (3.24)$$

where  $q_1$  is a constant, one finds

$$a_1 = (a_2 a_3)^{\mathcal{N}_2}, \quad \mathcal{N}_2 = (1 + 3q_1)/(2 - 3q_1). \quad (3.25)$$

Inserting this into (3.15) and using (3.16) one finally finds:

$$a_1 = V^{\mathcal{N}_2/(1+\mathcal{N}_2)}, \quad (3.26a)$$

$$a_2 = \mathcal{N}_1^{-1/(2n-m)} V^{(m-n-\mathcal{N}_2 m)/(\mathcal{N}_2+1)(m-2n)}, \quad (3.26b)$$

$$a_3 = \mathcal{N}_1^{1/(2n-m)} V^{(\mathcal{N}_2 m - n)/(\mathcal{N}_2+1)(m-2n)}. \quad (3.26c)$$

Let us note that perfect fluid satisfying the barotropic equation of state, as well as Chaplygin gas are described by a massless spinor field Lagrangian. Moreover,  $S$ , as well as  $F(S)$  are the functions of  $V$ . Therefore, we can rewrite the equations for defining  $V$  as

$$\ddot{V} = 2 \frac{m^2 - mn + n^2}{\mathcal{N}_1^{2/(2n-m)}} V^{q_2} + \mathcal{D}(V), \quad \mathcal{D}(V) = \frac{3\kappa}{2} \left[ S \frac{dF}{dS} - 2F \right] V, \quad (3.27)$$

with

$$q_2 = \frac{2(n - \mathcal{N}_2 m)}{(1 + \mathcal{N}_2)(m - 2n)} + 1 = \frac{1}{3} - \frac{2q_1(m + n)}{m - 2n}.$$

In case of spinor field nonlinearity given by (2.16), Eq. (3.27) takes the form

$$\dot{V} = 2 \frac{m^2 - mn + n^2}{\mathcal{N}_1^{2/(2n-m)}} V^{q_2} + \frac{3\kappa \nu C_0^{1+W} (1-W)}{2} V^{-W}, \quad (3.28)$$

with the solution in quadrature

$$\int \frac{dV}{\sqrt{q_3 V^{q_2+1} + 3\kappa \nu C_0^{1+W} V^{1-W} + C_1}} = t + t_0, \quad q_3 = \frac{4(m^2 - mn + n^2)}{(q_2 + 1) \mathcal{N}_1^{2/(2n-m)}}. \quad (3.29)$$

Here  $C_1$  is some integration constant.

Let us consider the case when the spinor field is given by the Lagrangian (2.21). The equation for  $V$  now reads

$$\dot{V} = 2 \frac{m^2 - mn + n^2}{\mathcal{N}_1^{2/(2n-m)}} V^{q_2} + \frac{3\kappa}{2} \left[ (AV^{1+\gamma} + \lambda C_0^{1+\gamma})^{1/(1+\gamma)} + \frac{AV^{1+\gamma}}{(AV^{1+\gamma} + \lambda C_0^{1+\gamma})^{\gamma/(1+\gamma)}} \right], \quad (3.30)$$

with the solution

$$\int \frac{dV}{\sqrt{C_1 + q_3 V^{q_2+1} + 3\kappa V (AV^{1+\gamma} + \lambda C_0^{1+\gamma})^{1/(1+\gamma)}}} = t + t_0, \quad C_1 = \text{const.} \quad t_0 = \text{const.} \quad (3.31)$$

Inserting  $\gamma = 1$  we come to the result obtained in (Saha, 2005).

Finally we consider the case with modified quintessence. In this case for  $V$  we find

$$\dot{V} = 2 \frac{m^2 - mn + n^2}{\mathcal{N}_1^{2/(2n-m)}} V^{q_2} + \frac{3\kappa}{2} \left[ \nu C_0^{1+W} (1-W) V^{-W} + 2W \epsilon_{\text{cr}} V / (1+W) \right], \quad (3.32)$$

with the solution in quadrature

$$\int \frac{dV}{\sqrt{q_3 V^{q_2+1} + 3\kappa [\nu C_0^{1+W} V^{1-W} + W \epsilon_{\text{cr}} V^2 / (1+W)] + C_1}} = t + t_0. \quad (3.33)$$

Recalling that in case of modified quintessence  $W \in (-1, 0)$ . So the model allows cyclic mode of expansion, only when  $q_2 < 1$ .

## 2. Bianchi type- $VI_0$ anisotropic cosmological model

Setting  $m = n$  from (3.1) we get Bianchi type- $VI_0$  cosmological model given by (3.2). In this case equation (3.14e) gives

$$\frac{\dot{a}_1}{a_1} - \frac{\dot{a}_2}{a_2} = 0, \quad (3.34)$$

with the solution

$$a_2 = \mathcal{N}_1 a_1, \quad (3.35)$$

Assuming that  $\sigma_1^1 \propto \theta$ , on account of (3.25) in this case we find

$$a_1 = V^{\mathcal{N}_2/(1+\mathcal{N}_2)}, \quad (3.36a)$$

$$a_2 = \mathcal{N}_1 V^{\mathcal{N}_2/(1+\mathcal{N}_2)}, \quad (3.36b)$$

$$a_3 = \frac{1}{\mathcal{N}_1} V^{(1-\mathcal{N}_2)/(1+\mathcal{N}_2)}. \quad (3.36c)$$

The equation for  $V$  in this case reads

$$\ddot{V} = 2m^2 \mathcal{N}_1^2 V^{q_2} + \mathcal{D}(V). \quad (3.37)$$

In this case for  $\text{VI}_0$  we find the solution in quadrature analogous to (3.29), (3.31) and (3.33), for quintessence, Chaplygin gas and modified quintessence, respectively, with

$$q_2 = \frac{2(\mathcal{N}_2 - 1)}{\mathcal{N}_2 + 1} + 1 = \frac{1}{3} + 4q_1, \quad q_3 = \frac{4m^2 \mathcal{N}_1^2}{q_2 + 1} = \frac{3m^2 \mathcal{N}_1^2}{1 + 3q_1}.$$

As we see, in presence of a modified quintessence,  $\text{BVI}_0$  cosmological model allows cyclic mode of evolution if  $q_1 < 1/6$ . In case of  $q_1 > 1/6$  we find the model is not bound from above as is seen from figure 3.

### 3. Bianchi type-V anisotropic cosmological model

Setting  $m = -n$  from (3.1) we get Bianchi type-V cosmological model given by (3.3). In this case equation (3.14e) gives

$$\frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} - 2\frac{\dot{a}_3}{a_3} = 0, \quad (3.38)$$

with the solution

$$a_1 a_2 = \mathcal{N}_1 a_3^2, \quad (3.39)$$

Assuming that  $\sigma_1^1 \propto \theta$ , on account of (3.25) in this case we find

$$a_1 = V^{\mathcal{N}_2/(1+\mathcal{N}_2)}, \quad (3.40a)$$

$$a_2 = \mathcal{N}_1^{1/3} V^{(2-\mathcal{N}_2)/3(1+\mathcal{N}_2)}, \quad (3.40b)$$

$$a_3 = \frac{1}{\mathcal{N}_1^{1/3}} V^{1/3}. \quad (3.40c)$$

The equation for  $v$  in this case reads

$$\ddot{V} = 6m^2 \mathcal{N}_1^{2/3} V^{q_2} + \mathcal{D}(V). \quad (3.41)$$

In this case for  $\text{V}$  we find the solution in quadrature analogous to (3.29), (3.31) and (3.33), for quintessence, Chaplygin gas and modified quintessence, respectively, with

$$q_2 = \frac{1}{3}, \quad q_3 = \frac{12m^2 \mathcal{N}_1^{2/3}}{q_2 + 1} = 9m^2 \mathcal{N}_1^{2/3}.$$

We can conclude that in presence of modified quintessence  $\text{BV}$  model always undergoes a cyclic mode of expansion.

#### 4. Bianchi type-III anisotropic cosmological model

Setting  $n = 0$  from (3.1) we get Bianchi type-VI<sub>0</sub> cosmological model given by (3.4). In this case equation (3.14e) gives

$$\frac{\dot{a}_1}{a_1} - \frac{\dot{a}_3}{a_3} = 0, \quad (3.42)$$

with the solution

$$a_3 = \mathcal{N}_1 a_1, \quad (3.43)$$

Assuming that  $\sigma_1^1 \propto \theta$ , on account of (3.25) in this case we find

$$a_1 = V^{\mathcal{N}_2/(1+\mathcal{N}_2)}, \quad (3.44a)$$

$$a_2 = \frac{1}{\mathcal{N}_1} V^{(1-\mathcal{N}_2)/(1+\mathcal{N}_2)}, \quad (3.44b)$$

$$a_3 = \mathcal{N}_1 V^{\mathcal{N}_2/(1+\mathcal{N}_2)}. \quad (3.44c)$$

The equation for  $V$  in this case reads

$$\ddot{V} = 2 \frac{m^2}{\mathcal{N}_1^2} V^{q_2} + \mathcal{D}(V). \quad (3.45)$$

In this case for  $V$  we find the solution in quadrature analogous to (3.29), (3.31) and (3.33), for quintessence, Chaplygin gas and modified quintessence, respectively, with

$$q_2 = -\frac{2\mathcal{N}_2}{\mathcal{N}_2 + 1} + 1 = \frac{1}{3} - 2q_1, \quad q_3 = \frac{4m^2}{(q_2 + 1)\mathcal{N}_1^2} = \frac{6m^2}{(2 - 3q_1)\mathcal{N}_1^2}.$$

As we see, in presence of a modified quintessence, BVI<sub>0</sub> cosmological model allows cyclic mode of evolution if  $q_1 > -1/3$ .

In the figures 1, 2, 3 and 4 we have plotted the evolution of  $V$  for different Bianchi models filled with quintessence and modified quintessence.

#### 5. Bianchi type-I anisotropic cosmological model

Bianchi type-I (BI) model is the simplest anisotropic cosmological model and gives an excellent scope to take into account the initial anisotropy of the Universe. Given the importance of BI model to study the effects of initial anisotropy in the evolution of the Universe, we study this models in details. Unlike the models considered previously, the system of Einstein's equations for BI model does not contain off-diagonal component (3.14e), explicitly relating metric functions between themselves. The system of Einstein equations in this case reads

$$\frac{\ddot{a}_2}{a_2} + \frac{\ddot{a}_3}{a_3} + \frac{\dot{a}_2 \dot{a}_3}{a_2 a_3} = \kappa T_1^1, \quad (3.46a)$$

$$\frac{\ddot{a}_3}{a_3} + \frac{\ddot{a}_1}{a_1} + \frac{\dot{a}_3 \dot{a}_1}{a_3 a_1} = \kappa T_2^2, \quad (3.46b)$$

$$\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1 \dot{a}_2}{a_1 a_2} = \kappa T_3^3, \quad (3.46c)$$

$$\frac{\dot{a}_1 \dot{a}_2}{a_1 a_2} + \frac{\dot{a}_2 \dot{a}_3}{a_2 a_3} + \frac{\dot{a}_3 \dot{a}_1}{a_3 a_1} = \kappa T_0^0. \quad (3.46d)$$

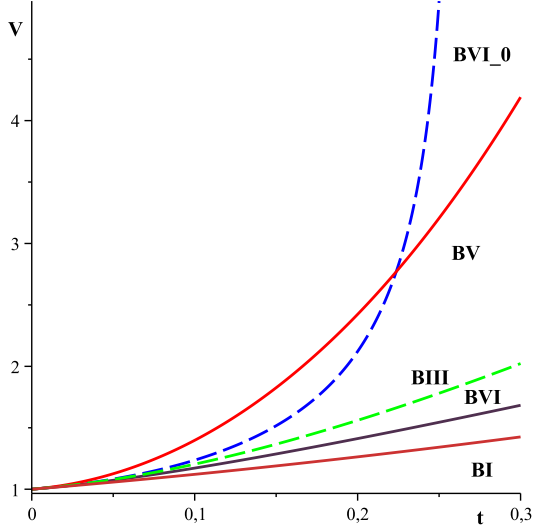


FIG. 1: Evolution of the Universe filled with quintessence for different Bianchi models

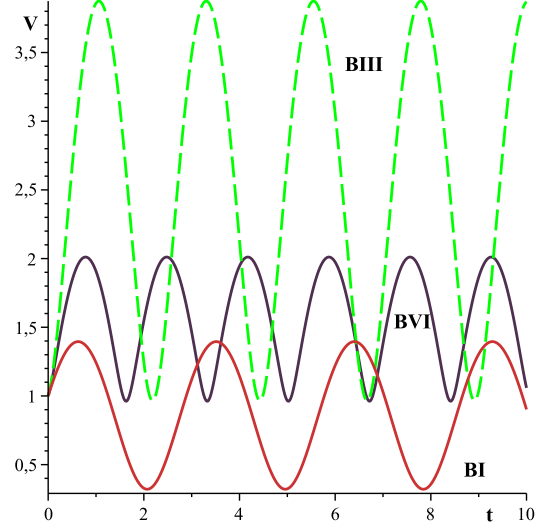


FIG. 2: Evolution of the Universe filled with modified quintessence for different Bianchi models

Solving the Einstein equation for the metric functions one finds (Saha, 2001a)

$$a_i = D_i V^{1/3} \exp\left(X_i \int \frac{dt}{V}\right), \quad \prod_{i=1}^3 D_i = 1, \quad \sum_{i=1}^3 X_i = 0, \quad (3.47)$$

with  $D_i$  and  $X_i$  being the integration constants.

The equation for  $v$  in this case takes the form (Saha, 2001a)

$$\ddot{V} = \mathcal{D}(V). \quad (3.48)$$

In case of (2.16) Eq. (3.48) takes the form

$$\ddot{V} = (3/2)\kappa v C_0^{1+W} (1-W) V^{-W} \quad (3.49)$$

with the solution in quadrature

$$\int \frac{dV}{\sqrt{3\kappa v C_0^{1+W} V^{1-W} + C_1}} = t + t_0. \quad (3.50)$$

Here  $C_1$  and  $t_0$  are the integration constants.

In the Figs. 5 and 6 we have plotted the evolution of the Universe defined by the nonlinear spinor field corresponding to perfect fluid and dark energy (Saha, 2010b).

Let us consider the case when the spinor field is given by the Lagrangian (2.21). The equation for  $V$  now reads

$$\ddot{V} = (3/2)\kappa \left[ (AV^{1+\gamma} + \lambda C_0^{1+\gamma})^{1/(1+\gamma)} + AV^{1+\gamma} / (AV^{1+\gamma} + \lambda C_0^{1+\gamma})^{\gamma/(1+\gamma)} \right], \quad (3.51)$$

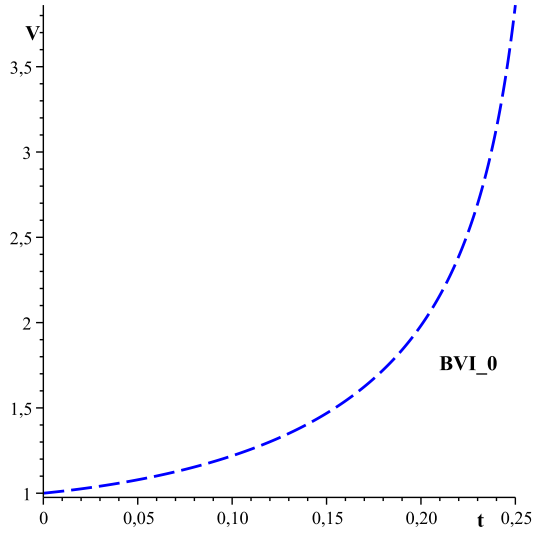


FIG. 3: Evolution of the Universe filled with modified quintessence for Bianchi type-VI<sub>0</sub> model

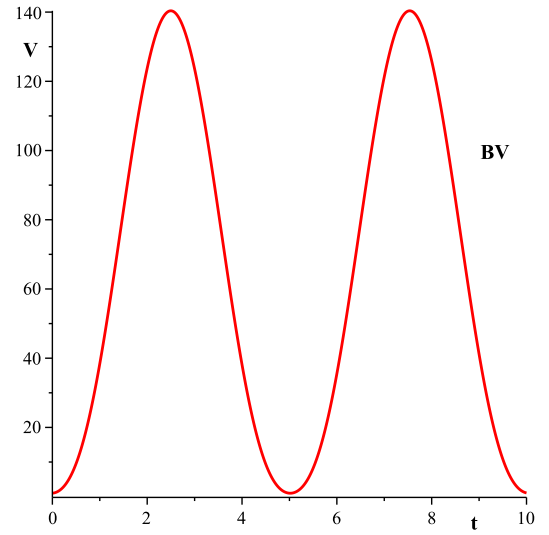


FIG. 4: Evolution of the Universe filled with modified quintessence for Bianchi type-V model

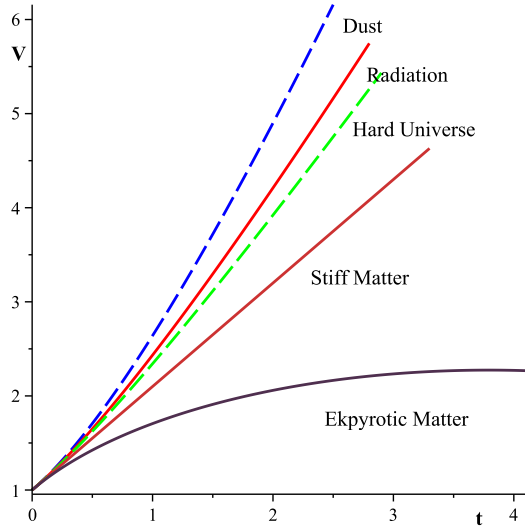


FIG. 5: Evolution of the Universe filled with perfect fluid.

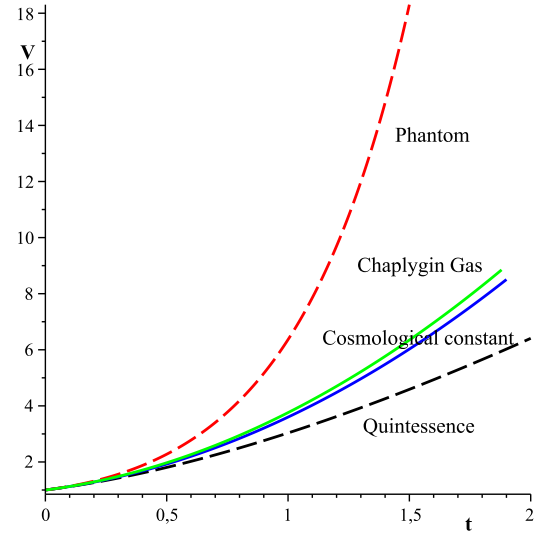


FIG. 6: Evolution of the Universe filled with dark energy.

with the solution

$$\int \frac{dV}{\sqrt{C_1 + 3\kappa V (AV^{1+\gamma} + \lambda C_0^{1+\gamma})^{1/(1+\gamma)}}} = t + t_0, \quad C_1 = \text{const.} \quad t_0 = \text{const.} \quad (3.52)$$

Inserting  $\gamma = 1$  we come to the result obtained in (Saha, 2005).

Finally we consider the case with modified quintessence. In this case for  $V$  we find

$$\dot{V} = (3/2)\kappa \left[ \eta C_0^{1-W} (1+W)V^W - 2W\epsilon_{\text{cr}}V/(1-W) \right], \quad (3.53)$$

with the solution in quadrature

$$\int \frac{dV}{\sqrt{3\kappa [\eta C_0^{1-W} V^{1+W} - W\epsilon_{\text{cr}} V^2/(1-W)] + C_1}} = t + t_0. \quad (3.54)$$

Here  $C_1$  and  $t_0$  are the integration constants. Comparing (3.54) with those with a negative  $\Lambda$ -term we see that  $\epsilon_{\text{cr}}$  plays the role of a negative cosmological constant.

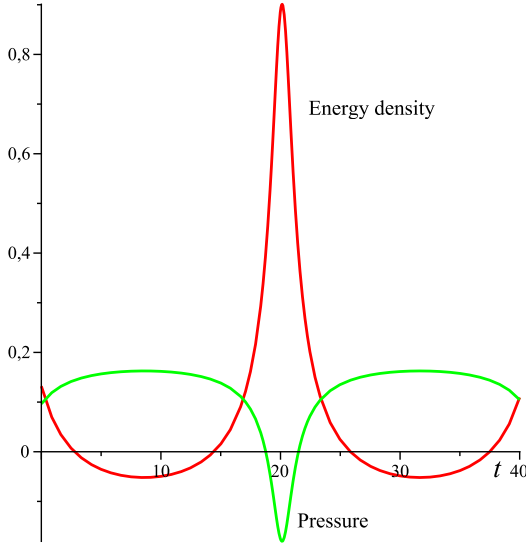


FIG. 7: Dynamics of energy density and pressure for a modified quintessence.

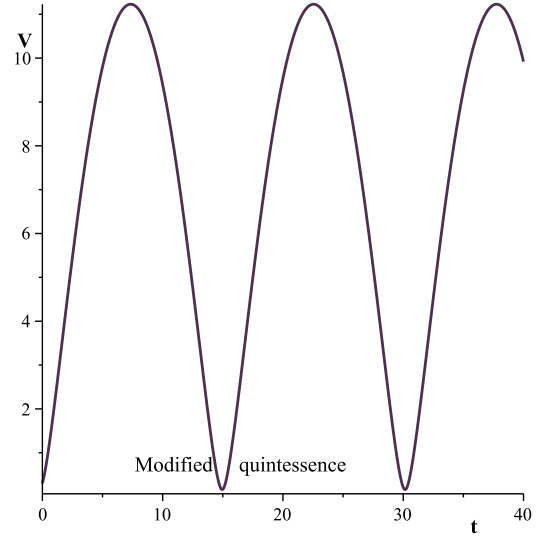


FIG. 8: Evolution of the Universe filled with a modified quintessence.

In the Fig. 7 we have illustrated the dynamics of energy density and pressure of a modified quintessence. In the Fig. 8 the evolution of the Universe defined by the nonlinear spinor field corresponding to a modified quintessence has been presented. As one sees, in the case considered, acceleration alternates with deceleration. In this case the Universe can be either singular (that ends in Big Crunch) or regular.

## 6. FRW cosmological models with a spinor field

Since our Universe is almost isotropic at large scale it would be fitting to study evolution of the FRW model within the scope of spinor description of matter. The Einstein equations read corresponding to the FRW model (3.6) reads

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = \kappa T_1^1 \quad (3.55a)$$

$$3\frac{\dot{a}^2}{a^2} = \kappa T_0^0. \quad (3.55b)$$

From the spinor field equations in this case we find

$$S = \frac{C_0}{a^3}, \quad C_0 = \text{const.} \quad (3.56)$$

In order to find the solution that satisfies both (3.55a) and (3.55b) we rewrite (3.55a) in view of (3.55b) in the following form:

$$\ddot{a} = \frac{\kappa}{6} (3T_1^1 - T_0^0) a. \quad (3.57)$$

Further we solve this equation for concrete choice of source field.

Let us consider the case of perfect fluid given by the barotropic equation of state. In account of (2.15a), (2.15b) and (3.56), (3.57) takes the form

$$\ddot{a} = \frac{\kappa v(1+3W)C_0^{1+W}}{2} a^{-(2+3W)}, \quad (3.58)$$

that admits the first integral

$$\dot{a}^2 = \frac{\kappa}{3} v C_0^{1+W} a^{-(1+3W)} + E_1, \quad E_1 = \text{const.} \quad (3.59)$$

In Fig. 9 and 10 we plot the evolution of the FRW Universe for different values of  $W$ .

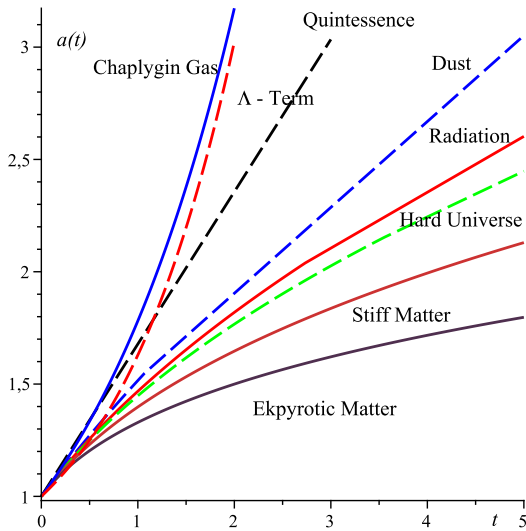


FIG. 9: Evolution of the Universe filled with perfect fluid and dark energy.

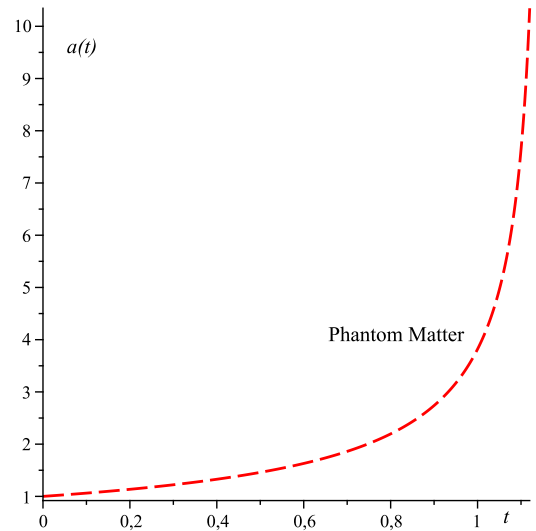


FIG. 10: Evolution of the Universe filled with phantom matter.

As one sees, equation (3.59) imposes no restriction on the value of  $W$ . But it is not the case, when one solves (3.55b). Indeed, inserting  $T_0^0$  from (2.15a) into (3.55b) one finds

$$a = (A_1 t + C_1)^{2/3(1+W)}, \quad (3.60)$$

where  $A_1 = (1+W)\sqrt{3\kappa v C_0^{1+W}/4}$  and  $C_1 = 3(1+W)C/2$  with  $C$  being some arbitrary constant. This solution identically satisfies the equation (3.55a). As one sees, case with  $W = -1$ , cannot



be realized here. In that case one has to solve the equation (3.55b) straight forward. As far as phantom matter ( $W < -1$ ) is concerned, there occurs some restriction on the value of  $C$ , as in this case  $A_1$  is negative and for the  $C_1$  to be positive,  $C$  should be negative. As one can easily verify, in case of cosmological constant with  $W = -1$  Eqn. (3.55b) gives

$$a = a_0 e^{\pm \sqrt{\kappa v/3} t}. \quad (3.61)$$

Inserting (2.20a) and (2.20b) into (3.57) in case of Chaplygin gas we have the following equation

$$\ddot{a} = \frac{\kappa}{6} \frac{2Aa^{3(1+\gamma)} - \lambda C_0^{1+\gamma}}{a^2 \left( Aa^{3(1+\gamma)} + \lambda C_0^{1+\gamma} \right)^{\gamma/(1+\gamma)}}. \quad (3.62)$$

We solve this equation numerically. The corresponding solution has been illustrated in Fig. 9.

Finally we consider the case with modified quintessence. Inserting (2.24a) and (2.24b) into (3.57) in this case we find

$$\ddot{a} = \frac{\kappa}{6} \left[ (3W - 1) \eta C_0^{1-W} a^{3W-2} - \frac{2W}{1-W} \epsilon_{\text{cr}} a \right], \quad (3.63)$$

with the solution

$$\dot{a}^2 = \frac{\kappa}{3} \left[ \nu C_0^{1-W} a^{3W-1} - \frac{W}{1-W} \epsilon_{\text{cr}} a^2 + E_2 \right], \quad E_2 = \text{const.} \quad (3.64)$$

It can be shown that in case of modified quintessence the pressure is sign alternating. As a result we have a cyclic mode of evolution.

#### IV. SINGULARITY PROBLEM

On the the main problems of modern cosmology is the singularity problem. Let us study this problem within the scope of models discussed above. As we see, the components of the spinor field and metric functions are expressed in terms of  $\nu$ . It should be noted that  $\nu$  plays one of the central roles in studying the singular space-time points of BI cosmological models. Here we describe it in brief. In doing so let us first write the Kretschmann scalar for the Bianchi type-VI metric. Taking into account that the metric (3.1) possesses the following non-trivial components of Riemann tensor:

$$\begin{aligned} R_{01}^{01} &= -\frac{\ddot{a}_1}{a_1}, & R_{02}^{02} &= -\frac{\ddot{a}_2}{a_2}, & R_{03}^{03} &= -\frac{\ddot{a}_3}{a_3}, \\ R_{12}^{12} &= -\frac{m\dot{n}}{a_3^2} - \frac{\dot{a}_1 \dot{a}_2}{a_1 a_2}, & R_{13}^{13} &= \frac{m^2}{a_3^2} - \frac{\dot{a}_3 \dot{a}_1}{a_3 a_1}, & R_{23}^{23} &= \frac{n^2}{a_3^2} - \frac{\dot{a}_2 \dot{a}_3}{a_2 a_3}, \\ R_{10}^{31} &= \frac{m}{a_3^2} \left( \frac{\dot{a}_1}{a_1} - \frac{\dot{a}_3}{a_3} \right), & R_{13}^{01} &= m \left( \frac{\dot{a}_3}{a_3} - \frac{\dot{a}_1}{a_1} \right), \\ R_{20}^{32} &= \frac{n}{a_3^2} \left( \frac{\dot{a}_3}{a_3} - \frac{\dot{a}_2}{a_2} \right), & R_{23}^{02} &= n \left( \frac{\dot{a}_2}{a_2} - \frac{\dot{a}_3}{a_3} \right), \end{aligned}$$

for the Kretschmann scalar we find

$$\begin{aligned} \mathcal{K} &= R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} = R^{\mu\nu}_{\alpha\beta} R^{\alpha\beta}_{\mu\nu} \\ &= 4 \left[ \left( \frac{\ddot{a}_1}{a_1} \right)^2 + \left( \frac{\ddot{a}_2}{a_2} \right)^2 + \left( \frac{\ddot{a}_3}{a_3} \right)^2 + \left( \frac{\dot{a}_1 \dot{a}_2}{a_1 a_2} \right)^2 + \left( \frac{\dot{a}_2 \dot{a}_3}{a_2 a_3} \right)^2 + \left( \frac{\dot{a}_3 \dot{a}_1}{a_3 a_1} \right)^2 \right] \\ &\quad + \frac{4}{a_3^4} \left[ (m^4 + m^2 n^2 + n^4) - (m^2 + n^2) \dot{a}_3^2 - \dot{a}_3^2 \left( m \frac{\dot{a}_1}{a_1} - n \frac{\dot{a}_2}{a_2} \right)^2 \right]. \end{aligned} \quad (4.1)$$

The metric functions for BVI, BVI<sub>0</sub>, BV and BIII can be expressed as

$$a_i = \mathcal{B}_i V^{s_i}, \quad (4.2)$$

which gives

$$\frac{\dot{a}_i}{a_i} = \mathcal{B}_i \frac{\dot{V}}{V}, \quad (4.3a)$$

$$\frac{\ddot{a}_i}{a_i} = \mathcal{B}_i \frac{\ddot{V}}{V} + \mathcal{B}_i (\mathcal{B}_i - 1) \left( \frac{\dot{V}}{V} \right)^2. \quad (4.3b)$$

The metric functions and their derivatives for the BI take the following form:

$$\frac{\dot{a}_i}{a_i} = \frac{\dot{V}}{3V} + \frac{X_i}{3V}, \quad (4.4a)$$

$$\frac{\ddot{a}_i}{a_i} = \frac{\ddot{V}}{3V} - \frac{2}{9} \left( \frac{\dot{V}}{V} \right)^2 - \frac{X_i}{9} \frac{\dot{V}}{V^2} + \frac{X_i^2}{9V^2}. \quad (4.4b)$$

We study the singularity on the basis of Kretschmann scalar and in doing so we follow the criteria given in (Bronnikov *et al*, 2004):

(i) For any finite  $t$  some  $a_i \rightarrow 0$ . (4.1) shows that if more than one scale factor becomes trivial at finite  $t$ , then it is a singularity. As is seen from (4.4) as  $V \rightarrow 0$  (i) all  $a_i \rightarrow 0$  if  $X_1 = X_2 = X_3 = 0$ , i.e., it is a singularity; (ii) more than one  $a_i \rightarrow 0$  if more than one  $X_i < 0$  and in this case we have singularity; (iii) only one  $a_i \rightarrow 0$  if only one  $X_i < 0$ , i.e., the space-time can be non-singular.

Note that this criteria of singularity always fulfills at the point where  $V = 0$ .

As far as  $t \rightarrow \infty$  is concerned, the corresponding asymptote can be singular if at least one  $a_i$  vanishes faster than exponentially. As  $X_1 + X_2 + X_3 = 0$ , it means one or more  $X_i$  is negative, and from (4.3) it follows that at least one function  $a_i$  vanishes faster than exponentially. Hence it is a singularity.

As far as BVI, BVI<sub>0</sub>, BV, BIII and FRW models are concerned, they are all singular at a space-time point, where  $V = 0$ . Moreover, in cases of BVI, BVI<sub>0</sub>, BV and BIII at least two scale factor  $a_i$ 's are directly related, hence there is always a possibility of more than one one scale factor becoming trivial, thus giving rise to a singularity at finite  $t$ .

Moreover, components of the spinor field as well as the physically observable quantities such as charge, current, spin etc. constructed from them are inverse functions of  $V$  (Saha, 2001a). Hence we conclude that within the scope of the model considered here at any point where  $V = 0$  there occurs a space-time singularity.

## V. PROBLEM OF ISOTROPIZATION

Since the present-day Universe is surprisingly isotropic, it is important to see whether our anisotropic BI model evolves into an isotropic FRW model. Isotropization means that at large physical times  $t$ , when the volume factor  $V$  tends to infinity, the three scale factors  $a_i(t)$  grow at the same rate. Two wide-spread definition of isotropization read

$$\mathcal{A} = \frac{1}{3} \sum_{i=1}^3 \frac{H_i^2}{H^2} - 1 \rightarrow 0, \quad (5.1a)$$

$$\Sigma^2 = \frac{1}{2} \mathcal{A} H^2 \rightarrow 0. \quad (5.1b)$$

Here  $\mathcal{A}$  and  $\Sigma^2$  are the average anisotropy and shear, respectively.  $H_i = \dot{a}_i/a_i$  is the directional Hubble parameter and  $H = \dot{a}/a$  average Hubble parameter, where  $a(t) = V^{1/3}$  is the average scale

factor. Here we exploit the isotropization condition proposed in Bronnikov *et al* (2004):

$$\left. \frac{a_i}{a} \right|_{t \rightarrow \infty} \rightarrow \text{const.} \quad (5.2)$$

Then by rescaling some of the coordinates, we can make  $a_i/a \rightarrow 1$ , and the metric will become manifestly isotropic at large  $t$ . It can be shown that in case of BVI, BVI<sub>0</sub>, BV and BIII models the criteria (5.2) does not hold, hence the isotropization process for these models does not take place. So we consider the BI model and study the problem of isotropization for this model in details.

From (3.47) we find

$$\frac{a_i}{a} = \frac{a_i}{V^{1/3}} = D_i \exp \left( X_i \int \frac{dt}{\tau} \right). \quad (5.3)$$

As is seen from (3.47) in our case  $a_i/a \rightarrow D_i = \text{const}$  as  $v \rightarrow \infty$ . Recall that the isotropic FRW model has same scale factor in all three directions, i.e.,  $a_1(t) = a_2(t) = a_3(t) = a(t)$ . So for the BI universe to evolve into a FRW one the constants  $D_i$ 's are likely to be identical, i.e.,  $D_1 = D_2 = D_3 = 1$ . Moreover, the isotropic nature of the present Universe leads to the fact that the three other constants  $X_i$  should be close to zero as well, i.e.,  $|X_i| \ll 1$ , ( $i = 1, 2, 3$ ), so that  $X_i \int [v(t)]^{-1} dt \rightarrow 0$  for  $t < \infty$  (for  $V(t) = t^n$  with  $n > 1$  the integral tends to zero as  $t \rightarrow \infty$  for any  $X_i$ ). It can be concluded that the spinor field Lagrangian with  $W < 1$  leads to the isotropization of the Universe as  $t \rightarrow \infty$ , moreover, in case of  $W < 0$  the system undergoes an earlier isotropization.

## VI. DISCUSSION

Let us now examine what kind of advantage one gets exploiting the spinor description of matter. We do it within the scope of a BI cosmological model. In doing so, we consider the case when the Universe is filled with, say, Van-der-Waals fluid, radiation and quintessence. In this case we have

$$T_0^0 = \varepsilon_v + \varepsilon_r + \varepsilon_q, \quad (6.1a)$$

$$T_1^1 = -p_v - p_r - p_q. \quad (6.1b)$$

To solve (3.27) one has to know  $T_0^0$  and  $T_1^1$  in terms of  $v$ . It can be done exploiting Bianchi identity

$$\dot{T}_0^0 + \frac{\dot{V}}{V} (T_0^0 - T_1^1) = 0. \quad (6.2)$$

The dark energy is supposed to interact with itself only, so it is minimally coupled to the gravitational field. As a result, the evolution equation for the energy density decouples from that of the perfect fluid. Taking this into account and inserting (6.1) into (6.2), we obtain two balance equations

$$\dot{\varepsilon}_v + \frac{\dot{V}}{V} (\varepsilon_v + p_v) + \dot{\varepsilon}_r + \frac{\dot{V}}{V} (\varepsilon_r + p_r) = 0, \quad (6.3a)$$

$$\dot{\varepsilon}_q + \frac{\dot{V}}{V} (\varepsilon_q + p_q) = 0. \quad (6.3b)$$

In usual approach we are in trouble, as neither  $\varepsilon_v$  nor  $\varepsilon_r$  cannot be expressed in terms of  $v$  from (6.3a). But if one uses spinor description for radiation, thanks to spinor field equation one finds

$$\dot{\varepsilon}_r + \frac{\dot{V}}{V} (\varepsilon_r + p_r) \equiv 0. \quad (6.4)$$

As a result, in place of (6.3a) we now have

$$\dot{\epsilon}_v + \frac{\dot{V}}{V}(\epsilon_v + p_v) = 0, \quad (6.5)$$

which is quite computable for the given equation of state. For Van-der-Waals fluid we use the following equation of state

$$p_v = \frac{8W_1\epsilon_v}{3 - \epsilon_v} - 3\epsilon_v^2, \quad (6.6)$$

where  $W_1$  is a constant. The dark energy density  $\epsilon_q$  can be expressed in terms of  $V$  either solving (6.3b) or using the spinor description. Thus using the spinor description (2.15a) and (2.15b) for radiation and dark energy, and defining the Hubble parameter, we find the following system of equations

$$\dot{V} = 3HV, \quad (6.7a)$$

$$\dot{H} = -3H^2 + \frac{\kappa}{2} \left[ \epsilon_v - \frac{8W_1\epsilon_v}{3 - \epsilon_v} + 3\epsilon_v^2 + \frac{2\epsilon_{r0}}{3}V^{-4/3} + (1 - W)\epsilon_{q0}V^{-1-W} \right], \quad (6.7b)$$

$$\dot{\epsilon}_v = -3 \left[ \epsilon_v + \frac{8W_1\epsilon_v}{3 - \epsilon_v} - 3\epsilon_v^2 \right] H. \quad (6.7c)$$

For simplicity, we set  $\epsilon_{r0} = 1$  and  $\epsilon_{q0} = 1$ . The equation (6.7c) can be exactly solved for  $W_1 = 1/2$ . In this case we find

$$\frac{\epsilon_v^6(3\epsilon_v - 7)}{(\epsilon_v - 1)^7} = \frac{C_0}{V^{14}}, \quad C_0 = \text{const.} \quad (6.8)$$

As one sees, it is quite a complicate expression. In what follows, we solve the system (6.7) numerically. For this purpose we also set  $W = -1/2$  for quintessence.

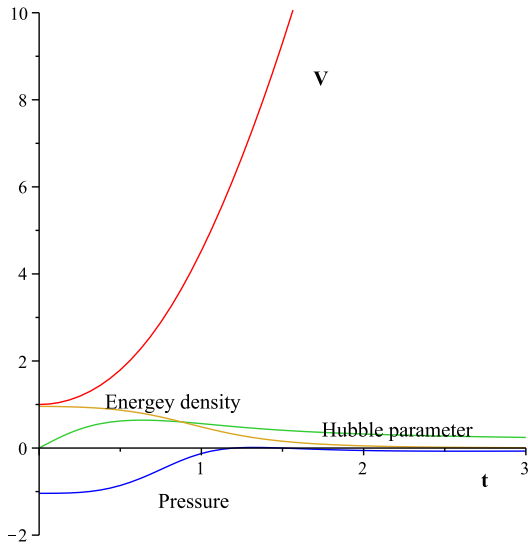


FIG. 11: Evolution of the  $V$ ,  $H$  and  $\epsilon_v$  and  $p$  for  $V(0) = 1$ ,  $H(0) = 0.0000945$  and  $\epsilon_v = 0.957$

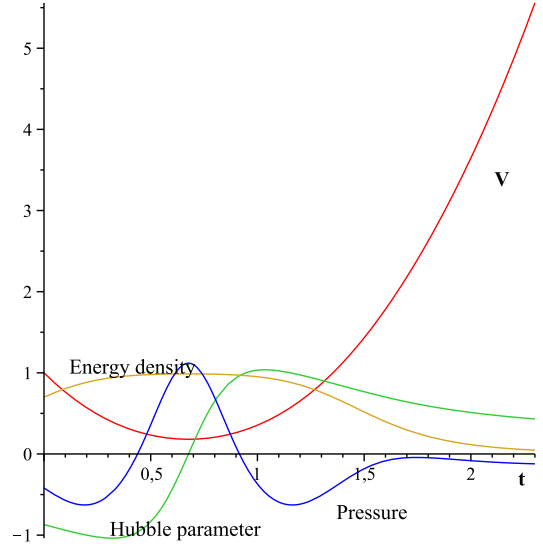


FIG. 12: Evolution of the  $V$ ,  $H$ ,  $\epsilon_v$  and  $p$  for  $V(0) = 1$ ,  $H(0) = -0.9$  and  $\epsilon_v = 0.7$

In the figures 11 and 12 we plot the evolution of  $V$ ,  $H$  and  $\epsilon_v$  and  $p$  for different initial values. As one sees, the pressure has negative value at the initial stage of evolution, then it begin to increase, thus giving rise to deceleration and finally becomes negative that results in the late time acceleration of the Universe.

## VII. CONCLUSION

Within the framework of cosmological gravitational field equivalence between the perfect fluid (and dark energy) and nonlinear spinor field has been established. It is shown that different types of dark energy can be simulated by means of a nonlinear spinor field. Using the new description of perfect fluid or dark energy evolution of the Universe has been studied within the scope of a BVI, BVI<sub>0</sub>, BV, BIII and BI anisotropic models as well as isotropic FRW model. The corresponding Einstein equations have been solved. It is shown that all the models give rise to a space-time singularity where  $V$  is trivial. Among the Bianchi models considered only the BI allows isotropization of the initially anisotropic space-time. It is shown that the spinor description of fluid allows one to solve the two-fluid system without without imposing any additional condition.

- 
- Armendáriz-Picón, C., Greene, P.B.: *Gen. Relat. Grav.* **35**, 1637 (2003).  
 Bento, M.C., Bertolami, O., Sen, A.A.: *Phys. Rev. D* **66**, 043507 (2002).  
 Bronnikov, K.A., Chudaeva, E.N., Shikin, G.N.: *Class. Quantum Grav.* **21**, 3389 (2004).  
 Cai, Y., Wang, J.: Dark energy model with spinor matter and its quintom scenario arXiv:0806.3890 [gr-qc] (2008).  
 Cardenas, R., Gonzalez, T., Leiva, Y., Martin, O., Quiros, I.: *Phys. Rev. D* **67**, 083501 (2003).  
 Dev, A., Alcaniz, J.S., Jain, D.: *Phys. Rev. D* **67**, 023515 (2003).  
 Fabris, J.C., Goncalves, S.V.B., de Souza, P.E.: *Gen. Relativ. Gravit.* **34**, 53 (2002).  
 Finkelstein, R., LeLevier, R., Ruderman, M.: *Phys. Rev.* **83**, 326 (1951).  
 Gross, D.J., Neveu, A.: *Phys. Rev. D* **10**, 3235 (1974).  
 Heisenberg, W.: *Physica* **19**, 897 (1953).  
 Heisenberg, W.: *Rev. Mod. Phys.* **29**, 269 (1957).  
 Henneaux, M.: *Phys. Rev. D* **21**, 857 (1980).  
 Kamenshchik, A.Yu., Moschella, U., Pasquier, V.: *Phys. Lett. B* **511**, 265 (2001).  
 Kibble, T.W.B.: *J. Math. Phys.* **2**, 212 (1961).  
 Krechet, V.G., Fel'chenkov, M.L., Shikin, G.N.: *Grav. & Cosmol.* **14** No 3(55), 292 (2008).  
 Misner, C.W.: *Asrophys. J.* **151**, 431 (1968).  
 Ribas, V.O., Devecchi, F.P., Kremer, G.M.: *Phys. Rev. D* **72**, 123502 (2005).  
 Saha, B., Shikin, G.N.: *Gen. Relat. Grav.* **29**, 1099 (1997a).  
 Saha, B., Shikin, G.N.: *J Math. Phys.* **38**, 5305 (1997b).  
 Saha, B.: *Phys. Rev. D* **64**, 123501 (2001a).  
 Saha, B.: *Mod. Phys. Lett. A* **16**, 1287 (2001b).  
 Saha, B., Boyadjiev, T.: *Phys. Rev. D* **69**, 124010 (2004a).  
 Saha, B.: *Phys. Rev. D* **69**, 124006 (2004b).  
 Saha, B.: *Chinese J. Phys.* **43**(6), 1035 (2005).  
 Saha, B.: *Astrophys. Space Sci.* **302**, 83 (2006a).  
 Saha, B.: *Int. J. Theor. Phys.* **45**(5) 983, (2006b).  
 Saha, B.: *Physics of Particles and Nuclei.* **37**. Suppl. 1, S13 (2006c).  
 Saha, B.: *Phys. Rev. D* **74**, 124030 (2006d).  
 Saha, B.: *Grav. & Cosmol.* **12**, N 2-3 (46-47), 215 (2006e).  
 Saha, B.: *Romanian Rep. Phys.* **59**, N 2, 649 (2007).  
 Saha, B.: *Cent. Euro. J. Phys.* **8**(6) 920 (2010a).  
 Saha, B.: *Romanian Rep. Phys.* **62**(1), 209 (2010b).

- Saha, B.: *Astrophys. Space Sci.* **331**(1), 243 (2011).  
de Souza, R.C., Kremer, G.M.: *Class. Quantum Grav.* **25**, 225006 (2008).  
Zlatev, I., Wang, L., Steinhardt, P.J.: *Phys. Rev. Lett.* **85**, 896 (1999).